

Mathematical Modeling in Digital Environment and its Implications on Teaching and Learning Extreme Values and Functions' Monotonicity

Tanja Sekulić ^{1*}, Valentina Kostić ²

¹ Technical College of Applied Sciences in Zrenjanin, Zrenjanin, Serbia

² Academy of Applied Technical and Preschool Studies Niš - Department Pirot, Serbia

* tsekulicvts@gmail.com

Abstract: *In this paper, we present a way of introducing teaching and learning method based on the principles of mathematical modeling empowered by digital technologies. The role of mathematical modeling in educational practice is discussed as also the implications of the realization of mathematical modeling in a computer environment. In the example of teaching Extreme values and functions' monotonicity, step by step procedure of making a mathematical model is explained – from starting preparation, to implementing GeoGebra as the software tool. The experiences of students and teachers during and after learning and teaching using mathematical modeling in the GeoGebra environment were supportive concerning further implications of this method.*

Keywords: *Mathematical modeling, GeoGebra, Extreme values, Functions' monotonicity*

1. INTRODUCTION

Computer technologies are present in education for quite some time. However, recently emerged new conditions and restrictions concerning pandemics, which imposed new demands on teachers, students, and the educational process itself.

The teaching methods had to be adaptable not only to the classroom but also to the online and hybrid teaching and learning process. The teaching materials had to be adapted in order to, in some way, simulate the real conditions and to have visual and dynamical elements with which could be manipulated and experimented.

For that purpose, we decided to use mathematical modeling, especially realized in a computer environment, mostly because of our previously good experiences with it [1].

On the example of teaching Extreme values and functions' monotonicity, we described the application of mathematical modeling realized in GeoGebra environment. The modeling process and the use of GeoGebra options are described in detail, as also the reactions of students.

2. MATHEMATICAL MODELING

Mathematical modeling is a mapping between reality and mathematics. The purpose of the mathematical model is reflected in drawing conclusions about reality based on the model. The most important for a mathematical model is that it

represents a basis for drawing conclusions about reality which can then be tested experimentally.

2.1. Mathematical modeling in education

The role of mathematical modeling is very significant, viewed from the standpoint of education. The need for the application of mathematical modeling in the educational process is mostly reflected in the fact that students leave schools and colleges with a relatively high level of theoretical knowledge, but are not adequately prepared for the application of that same knowledge in practice. The modern business and economic interests of employers require students to be prepared and trained to solve problems from real practice, which means that they must be ready at all times to adapt to new situations and the tasks they entail. This means that those from schools and colleges must leave as better-trained experts.

Taking into account that mathematical modeling is a technique of teaching based on connecting real-life problems with their mathematical representation, it can be considered a tool for illustrating mathematical content and for motivating students for learning and application of gained knowledge.

2.2. Mathematical modeling process

Establishing a relationship between mathematics and the real world is actually a process of mathematical modeling. The modeling process is not rather complicated and there are

different approaches and points of view concerning the modeling phases, i.e. modeling cycle [2].

The most common mathematical modeling cycle consists of several phases, which are: selection of a real-world situation that will be modeled, based on the selected real situation the real-world problem is observed, a mathematical model based on the real-world problem is made and a mathematical solution is derived from the mathematical model. If the solution is accepted, a report about it is made. If not, we can do a revision of the modeling process, and the mathematical model itself.

The modeling processes used for educational purposes are constantly improved, especially good results are achieved by combining mathematical modeling with computer technologies.

3. DIGITAL ENVIRONMENT AND MATHEMATICAL MODELING

Due to the complexity of the mathematical modeling process, there is a general opinion that it should be realized with the application of computer technologies [3]. It is considered that the best results are obtained when mathematical modeling is combined with the application of computer technologies because computers add a visual dimension. The contribution of computer technologies is reflected in the ability to quickly generate a large number of examples that students can explore and examine the relationships that exist between them. Working with dynamic displays with the use of interactive tools is even more effective because it gives students the opportunity to manipulate mathematical objects, which greatly facilitates the process of mathematical modeling.

3.1. GeoGebra as a digital tool for mathematical modeling

In this paper, we presented the use of GeoGebra software as the support for the modeling process. GeoGebra is dynamical mathematical software that can be used for implementing various mathematical theories. The main characteristic of GeoGebra, when considering its application for educational purposes, is its simplicity of use, and low teacher training demands. But nevertheless, this is very powerful software, with the possibility to respond to all requests of the user, especially when used in the mathematical modeling process. The dynamical nature of GeoGebra is one of the main reasons for being the most used software in the modeling process because GeoGebra can describe real-life problems through animations and simulations, and at the same time build a mathematical representation of the considered problem.

Also, we used GeoGebra because it enables a direct connection between the visual/dynamical contents

and its mathematical representation. In that way, students are becoming familiar with formal mathematics' language and at the same time, they can see how it is connected with the meaning of the real-life problem and how it represents mathematically.

Today, GeoGebra represents software that is constantly present at all levels and in many areas of education, especially in the teaching of mathematics and natural sciences [4]. The positive results of using GeoGebra have been confirmed in the teaching and learning of mathematics at all levels of education which is one of the main reasons why this software was chosen to implement the modeling process [1], [5].

4. LEARNING EXTREME VALUES AND FUNCTIONS' MONOTONICITY USING MATHEMATICAL MODELING AND GEOGEBRA

The application of mathematical modeling using computer technologies and GeoGebra will be demonstrated on the example of teaching and learning extreme values and functions' monotonicity.

4.1. The modeling process

To describe the real-world situation that was used for modeling, a part of a popular cartoon was used in which its main character moves up and down and during that movement passes through positions corresponding to his highest/lowest achieved height. The story that was used for the real-world situation is based on a famous cartoon (more precisely one of its parts from 1 min 40 s to 2 min 20 s) where the main character Coyote the Genius (wolf) tries to catch the Road Runner Bird (Wile E. Coyote and The Road Runner–Beep Beep, 1952).

First, the teacher showed the cartoon to the students on YouTube:

(<https://www.youtube.com/watch?v=132XmBjsOiE>).

The wolf in the cartoon is trying to cross a wire strung between two hills, while holding an anvil in his hands. Due to the weight of the anvil, the wire stretches, lowering the wolf to the ground. The moment the wolf touches the ground, it drops the anvil and, since it is on a wire, is launched into the air. For a certain period of time, it flies up through the air, slows down, reaches a maximum, and then begins to fall down. In the end, because he fails to open the parachute, the wolf falls to the ground, (Fig. 1).

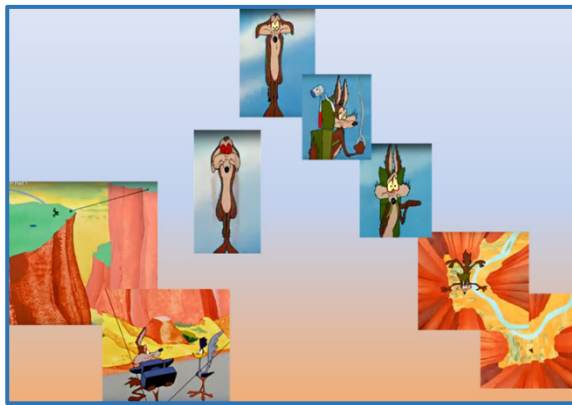


Figure 1. Cartoon story

After the students saw the cartoon, the teacher posed the real problem that corresponds to the situation being modeled.

The real problem: Model the movement of a character from a cartoon, that is, mathematically represent and describe his movement.

The students concluded that they should first find a function that corresponds to the two-dimensional representation of the wolf's trajectory. However, many students commented that the cartoon did not provide them with any information that could help them obtain the required function. The teacher then, in order to help the students, said that they could observe the function on the interval $[0,10]$ and assume the following: that the wolf at the beginning of its movement was at the position corresponding to the coordinates $(0,8)$, after for a few moments it was at the position with coordinates $(2,0)$, and then passed through the point $(5,9)$. It completed its movement when it fell to the ground, at the point $(10,0)$. The students used GeoGebra, they imported points A, B, C and D into GeoGebra and applied the GeoGebra option *Polynomial*. In this way, they came up with a mathematical model of the wolf's path (Fig. 2).

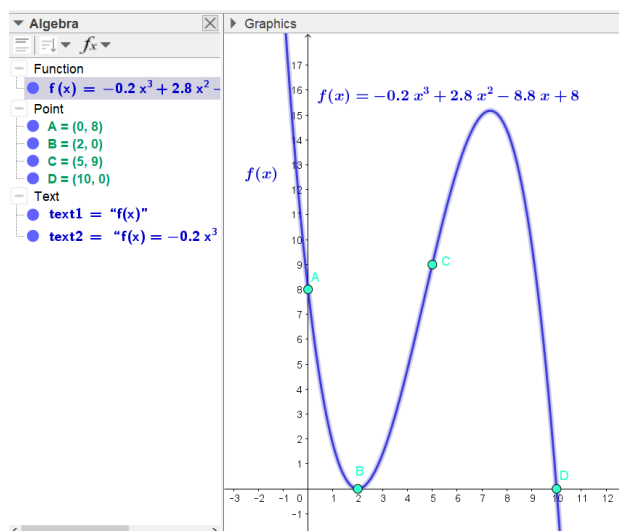


Figure 2. The mathematical model of the path

Continuing the modeling process, the students added images of individual frames from the cartoon to the GeoGebra material they had created up to that point, in order to additionally illustrate the movement of the character, (Fig. 3).

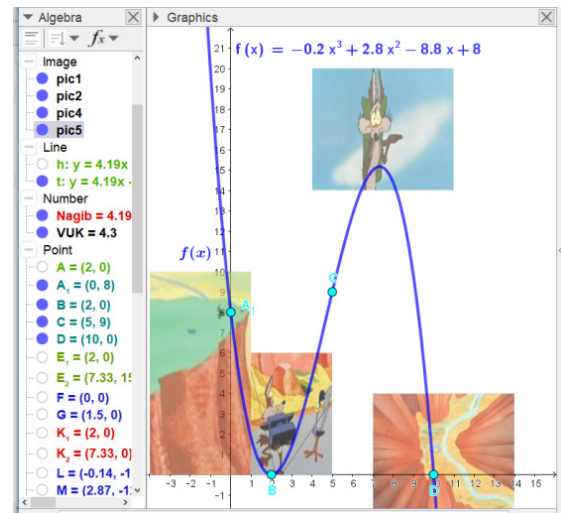


Figure 3. Illustrated path

In order to further model and experiment with the movement of the wolf, the students continued the development of the GeoGebra material by using the opportunity to add another window, activating the GeoGebra options *View*→*Graphics 2*. In the second window, they extracted the path function f and limited its display to the observed interval $[0,10]$ because they wanted to observe only the movement of the wolf from the starting point $A(0,8)$ to the point $T(10,0)$ when they considered that it finally fell to the ground and completed its movement, (Fig. 4). The point $T(10,0)$ was obtained by the students using the GeoGebra option *Intersect* (intersection of the function f and the Ox axis).

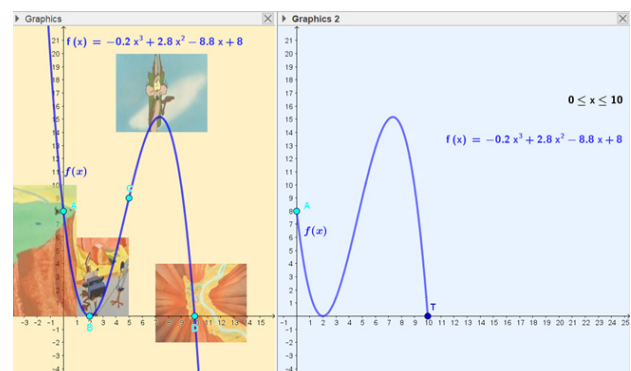


Figure 4. Dual window representation of the path function

The students then observed what happens to the slope when the wolf moves along the function f . For this purpose, they added a GeoGebra option *Slider* (marked with VUK) on the second graphic,

the purpose of which was to move the point that would represent the wolf according to the function f . Also, they tied the tangent t on the function f to the slider at the point representing the wolf and turned on the *Slope* option for the slope of the tangent, and displayed everything within the first and second graphs, (Fig. 5).

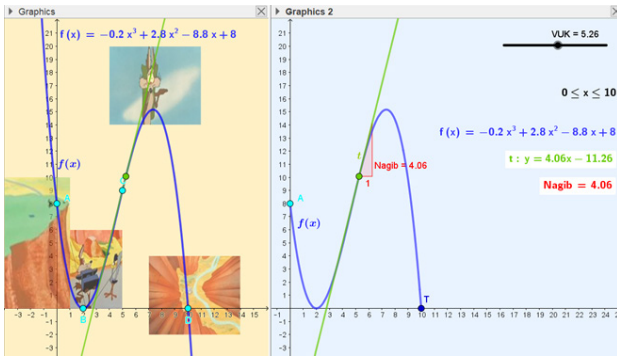


Figure 5. The movement of the wolf according to the function f , the tangent t , and its slope at the points of its positions

By moving the slider, the students could observe the movement of the cartoon character and the slope values. In that way, they noticed that the slope of the tangent to the function f is positive when the wolf is moving up, and when it is moving down, the slope has a negative value.

Next, they used GeoGebra's Derivative option to draw the derivative function from the function f and then observed how the changing sign of the slope affects the derivative function f' , (Fig. 6).

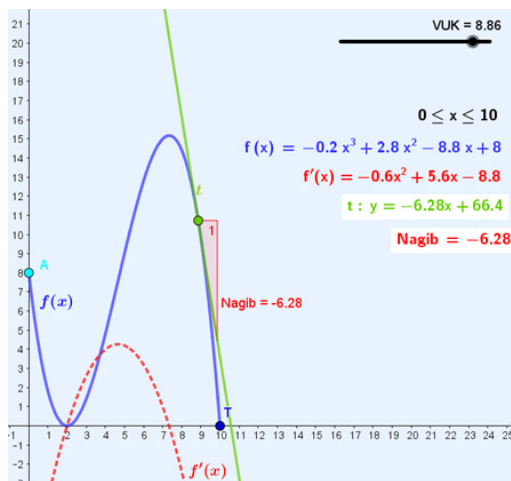


Figure 6. The derivative function f' and the sign of the slope of the tangent t

By comparing the sign of the slope of the tangent t to the function f and the sign of the derived function f' , after a short experiment with the GeoGebra material they created, the students concluded that the positive sign of the slope of the tangent t appears when the wolf moves up, i.e.

when the sign of the derived function is f' is positive and vice versa (negative slope – the wolf moves down – the sign of the derivative function is negative).

Further, the students used GeoGebra again and the *Intersect* option, which they now applied to the derived function f' and the Ox axis. From GeoGebra, the students read the coordinates of the intersection points of the derivative function f' and the Ox axis: $K_1(2,0)$ and $K_2(22/3,0)$, (Fig. 7).

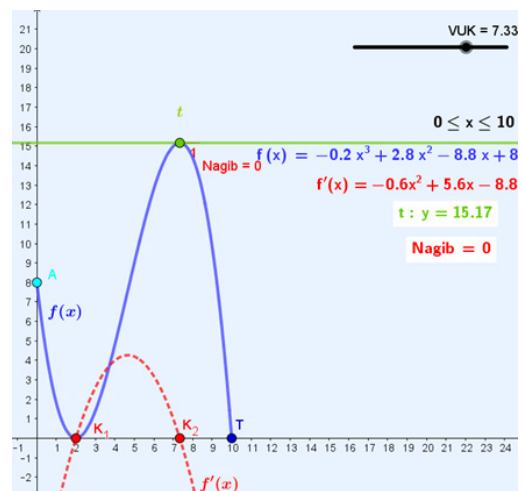


Figure 7. Zeroes of the derivative function f' (intersection of the derivative function f' and the Ox axis)

Observing the points K_1 and K_2 and the subintervals they determine on the interval $[0,10]$ and analyzing the sign of the derived function, the students came to a part of the mathematical solution to the real problem, that is, they could determine the intervals in which the wolf moves up/ down.

Furthermore, the students observed the function f and the slope of its tangents near the points $(2, f(2))$ and $(22/3, f(22/3))$. They noticed that the slope of the tangent of the function f changes its sign near the observed points (that is, the wolf goes up to each of the observed points and then down, or vice versa). Then they entered the mentioned points into the GeoGebra material, and by analyzing the graphic representation of the function f concluded that these points correspond to its extreme values: the minimum (point E_1) and the maximum (point E_2) on the observed interval $[0,10]$, (Fig. 8).

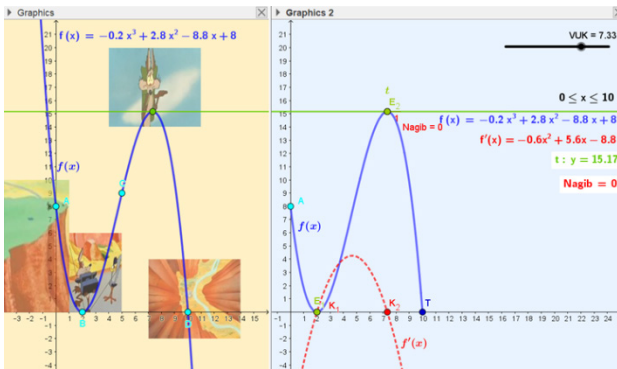


Figure 8. Extreme values of function f and slope of the tangent at point E_2

Connecting all the conclusions reached during the previous modeling process related to the up/down movement of the wolf, the sign of the derived function f' and its zeroes, as well as the tangents to the function f at its points, the students came to the mathematical solution of the situation from the real world they have modeled. The mathematical solution to the posed real problem is presented in Table 1.

Table 1. The mathematical solution of the real problem

Interval	$[0, 2)$	$(2, \frac{22}{3})$	$(\frac{22}{3}, 10]$
The sign of the function f'	-	+	-
Functions monotonicity	↘	↗	↘
Extreme value	$E_1(2, 0)$ <i>min</i>	$E_2(\frac{22}{3}, f(\frac{22}{3}))$ <i>max</i>	

The mathematical model and mathematical solution were accepted after being previously validated in GeoGebra.

5. REMARKS AND DISCUSSION

The presented process of mathematical modeling in the GeoGebra environment illustrates the work with students in the processing of the topic: Extreme values and functions' monotonicity. The impressions of the students during and after the modeling process were positive, emphasizing that this type of teaching and learning suits them much more than the classical one, in the sense that connecting the teaching content with real situations enables better learning and understanding, which actually achieved the goal of the application of the mathematical modeling.

The students marked one detail as very useful and interesting, the possibility to use the material in

their own time and to access it multiple times. In their words, they could practice and experiment with different functions and by it, learn the application of the functions' monotonicity and the nature of extreme values.

The teachers also had very positive experiences with this material, and it was used for teaching in a few high schools by different teachers who all agreed that this material was of great help in realizing the teaching process. The teachers also commented that the results of students' achievements were at a satisfactory level during the online teaching process and connect these good results with the application of mathematical modeling and GeoGebra.

These kinds of materials, based on both, mathematical modeling and realized in GeoGebra, are to be even more present in the teaching and learning process, online and in the classroom, because of the good experiences of teachers who used them and on demand of our students.

6. CONCLUSION

Finding the most appropriate way of teaching which will lead to optimal learning results can be very challenging, especially if one of the goals is to transfer functional knowledge to our students.

Also, the recent conditions concerning pandemics have greatly influenced to change and adaptation of educational methods.

Mathematical modeling proved to be a highly efficient method for teaching mathematics and sciences precisely because of its feature to connect real-world situations with its mathematical representations. The digital environment has intensified the modeling capabilities as it enabled faster and simpler manipulation of objects and added a dynamic effect through simulations and animations.

The students accepted mathematical modeling as a teaching and learning method and their impressions are more than positive. The teachers also confirmed the benefits of modeling, concerning both, the easier realization of the teaching process and the better achievements of students.

Future plans include the further adaptation of the curriculums to the application of mathematical modeling and sharing good practice with our colleagues.

REFERENCES

- [1] Sekulic, T., Takaci, D., Strboja, M., & Kostic, V. (2020). Influence of Mathematical Modeling in GeoGebra Environment on Learning Derivative. *The International Journal for Technology in Mathematics Education*, vol. 27, no. 2, 61-82. doi: 10.1564/tme_v27.2.01

- [2] Galbraith, P. L., Stillman, G., & Brown, J. (2010). Turning Ideas into Modeling Problems. Modeling Students' Mathematical Modeling Competencies, *ICTMA 13*, Section 4, New York: Springer, 133-144.
- [3] Valdés y Medina, E. G., & Medina Valdés, L. (2015). Dynamic Models as change Enablers in Educational Mathematics. *Procedia – Social and Behavioral Sciences*, Volume 176, 923-926.
- [4] Kostić, V., Stankov Jovanović, V., Sekulić, T. and Takači, Dj. (2016). Visualization of problem solving related to the quantitative composition of solutions in the dynamic GeoGebra environment. *Chemistry Education Research and Practice*, vol. 17, no. 1, 120-138. doi: 10.1039/c5rp00156k
- [5] Sekulić, T., Manigoda, G., Kostić, V. (2022). Teaching Trigonometry Online Using Geogebra Dynamical Materials. *Sinteza 2022 - International Scientific Conference on Information Technology and Data Related Research*, 121-128. doi:10.15308/Sinteza-2022-121-128